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Combinatorial Structures on Triangulations. II. Local Colorings

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1. VARIOUS TYPES OF COLORING

On the sphere there are many ways of "coloring" which are equivalent to four coloring [3]. In this section we will study the relationships between three of these—four coloring, edge coloring, and heawood coloring—on an arbitrary orientable two manifold. We introduce the concept of a local coloring which clarifies the relations.

(1) A *four coloring* of a triangulation K is a map $f: K \to \partial\Delta^3$ which is simplicial and maps triangles onto triangles. We call any map between 2-complexes with these two properties a nondegenerate map. $\partial\Delta^3$ is the tetrahedron, so the vertices of K are mapped to four vertices. If two vertices of K are adjacent, then the edge between them is mapped to an edge, so they map to different vertices. Thus this definition is the same as the usual one.

(2) An *edge coloring* of K is a division of the edges of K into 3 disjoint sets, so that every triangle has an edge in each set. One usually thinks of the sets as 'colors', so the edges of every triangle are 3-colored.

(3) A *heawood coloring* is an assignment of $+1$ or -1 to each triangle of a triangulation K such that the sum of the values on all triangles containing a vertex p is zero mod 3 for all vertices p of K.

(4) We give the formal definition of a local coloring now. As we go on, it will become clear how it relates to the other types of coloring.

A *local coloring* is a collection of maps $f_p: st(p) \to \partial\Delta^3$, one for each vertex p, and of automorphisms σ_{pq} of the tetrahedron (there are 24 possible), one for each pair of vertices p and q, such that the following diagram commutes:

$$\begin{tikzcd} & st(p) \cap st(q) \arrow[dl, "f_p"'] \arrow[dr, "f_q"] & \\ \partial\Delta^3 \arrow[rr, "\sigma_{pq}"'] & & \partial\Delta^3 \end{tikzcd}$$

Here $St(p)$ is the union of all triangles containing p. We can see this as saying that we have a four coloring for each $St(p)$, and when a $St(p)$ and $St(q)$ overlap, the colorings agree—up to a permutation of colors.

For example, suppose K is a triangulation such that every vertex has even degree. Then there is a canonical local coloring, called the local 3-coloring of K, defined as follows: the degree of p is even, so there is a unique map (up to permutation of colors) $f_p \colon st(p) \to \Delta^2$. Given these f_p, it is easy to find the σ_{pq}.

There is a more convenient representation for local colorings than the (f_p, σ_{pq}) presentation. Suppose we have a map $f \colon K \to \partial\Delta^3$, and e is an edge of K. We call e a *nonsingular* edge of f if f maps the two triangles containing e to distinct triangles. We call e *singular* if f maps the two triangles on to the same triangle. If σ is an automorphism of $\partial\Delta^3$, then an edge e is nonsingular for f iff it is nonsingular for σf. Thus we can talk about an edge of a local coloring being singular or nonsingular. It is easy to see that an assignment of edges as singular or nonsingular arises from at most 1 local coloring. For example, in Fig. 1 we have a local coloring of the unique triangulation of the torus with 7 vertices. The nonsingular edges are the darker edges, and the edges of the boundary of the square are not part of the triangulation:

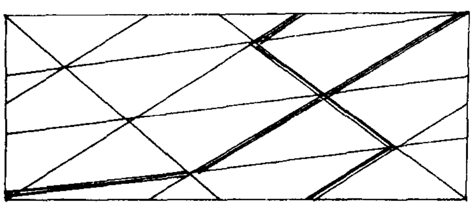

FIGURE 1

At the unique vertex p with four nonsingular edges, the local coloring is given as follows: We think of $\partial\Delta^3$ having vertices 0, 1, 2, 3. Then f_p of vertex 6 is 0, of vertices 3 and 2 is 1, of vertices 5 and 4 is 2, and of vertices 1 and 7 is 3. Notice that this is not a four coloring outside of $St(p)$.

Given a random assignment of edges into classes "nonsingular" and "singular," it probably doesn't come from a local coloring. For instance, the assignment of every edge as singular corresponds to a local coloring iff every vertex has even degree. If we take the assignment where every edge is nonsingular, then this corresponds to a local coloring iff every vertex has degree divisible by 3.

We shall now discuss how one type of coloring induces other types. Suppose we have a four coloring f of a triangulation K. We obtain an edge coloring of K by observing that $\partial\Delta^3$ has a unique edge coloring, and pulling it back to K by f. That is, if e is an edge of K, the 'color' of e is the 'color' of $f(e)$.

Next, suppose α is an edge coloring of K. Let the 3 colors be a, b and e. If a triangle of K is colored a, b, c (in the positive direction) we assign the triangle $+1$. If it's assigned a, c, b we assign it -1. To show that this is a heawood coloring is a local problem. We summarize this and similar local problems in the

LEMMA 1. *In* $St(p)$:

(1) *an edge coloring is induced by a* 4 *coloring*;

(2) *a heawood coloring is induced by a* 4 *coloring*;

(3) *an edge coloring induces a heawood coloring.*

Proof. For (1), color a vertex of the boundary of $St(p)$ [written $\text{ln}\, k(p)$] the color of the edge joining it to p. Assigning p the fourth color gives us a four coloring which is easily seen to induce the edge coloring.

For (2), let the vertices of $\partial\Delta^3$ be 0, 1, 2, 3. Map p to 3. Choose a vertex q on $\text{ln}\, k(p)$ and map it to 0. If we have defined our four coloring f on a vertex τ, and s is adjacent to r, put $f(s) = f(\tau) +$ assignment to the triangle containing rs, where addition is taken mod 3. The condition that the sum of the assignments is zero mod 3 around p guarantees that f will be well defined.

For (3), we use (1) to get a four coloring. Then a similar argument to (2) gives us the result. Q.E.D.

Since locally a heawood coloring is induced by a four coloring, we easily see that a heawood coloring induces a local coloring.

It is not true that the converses of these implications hold. For instance, if we take the local 3 coloring of the triangulation in Fig. 1, we can see it is induced by a heawood coloring, but not by an edge coloring. The local coloring given in Fig. 1 is not induced by a heawood coloring. If we assign all the edges to be nonsingular, then that local coloring is induced by an edge coloring but not by a four coloring. Indeed, the figure has no four colorings, but has 40 different local colorings.

From the formal definition of local coloring, we see that we have a fiber bundle with structure group $S(4)$, the symmetric group on 4 letters. Consequently, we have the characteristic homomorphism:

$$\psi(\alpha): \Pi_1(M) \to S(4)$$

defined for every local coloring α. We get this directly as follows: let γ be a path representing an element of $\Pi_1(M)$. If we start with a map $g: \Delta \to \partial\Delta^3$ where Δ is a triangle of M meeting γ, then using the local coloring, there is a unique extension of g to a neighborhood of Δ. We continue extending g along a neighborhood of γ until we arrive at Δ again. We thus have maps g and $\hat{g}$. There is a unique element σ of $S(4)$ such that $\sigma g = \hat{g}$. We put $\psi(\alpha)(\gamma) = \sigma$.

$\psi(\alpha)$ measures the obstruction to one type of coloring being reduced by another, as we see in the following

THEOREM 2. *Let α be a local coloring of M. Then*

(1) *α is induced by a four coloring iff* $\mathrm{Im}[\psi(\alpha)] = Id$.

(2) *α is induced by an edge coloring iff* $\mathrm{Im}[\psi(\alpha)] \subseteq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

(3) *α is induced by a heawood coloring iff* $\mathrm{Im}[\psi(\alpha)] \subseteq A(4)$.

Although $\psi(\alpha)$ is only well defined up to conjugacy, the notion of $\mathrm{Im}\,\psi(\alpha) \subseteq N$ is well defined, if N is a normal subgroup. The groups appearing on the right of the theorem are all the normal subgroups of $S(4)$.

We will prove (2) the others being similar. We observe that $Z_2 \oplus Z_2$ is the subgroup of $S(4)$ which fixes a given edge coloring of $\partial\Delta^3$. We define the edge coloring inducing α by defining it along a path as we did for the definition of $\psi(\alpha)$. We get a unique edge coloring on any closed path, since the permutation we get on the local coloring fixes our edge coloring.

We see the group $A(4)$ as the Subgroup of $S(4)$ which fixes a heawood coloring. It is all orientation preserving automorphisms of $\partial\Delta^3$.

2. Stieffel–Whitney Class of a Local Coloring

In this section we study the obstruction to a local coloring being induced by a heawood coloring. If α is a local coloring, then we just saw that α is induced by a heawood coloring iff $\mathrm{Im}[\psi(\alpha)] \subseteq A(4)$. By taking quotients, we can restate this as follows:

α is induced by a heawood coloring iff

$\psi'[\alpha]\colon \Pi_1(M) \to S(4)/A(4) \approx Z_2$ is the zero map.

Since H_1 is the commutator of Π_1, and Z_2 is abelian we get a map from $H_1(M)$ to Z_2, which we also call ψ'. The map $\psi'(\alpha)$ has the property that $\psi'(\alpha) = 0$ iff α is induced by a heawood coloring.

We now give a local computation of $\psi'(\alpha)$ on elementary cycles of $H_1(M)$. By an elementary cycle x of $H_1(M)$ we mean that x has at most two edges in common with any vertex. x therefore consists of disjoint simple closed curves, together with an orientation. Define $p_+(p, x)$ to be 0 if p is not on x. If p is on the cycle x, consider one side of the cycle x, say the positive side. It's meaningful to do this, for M and X are oriented. Let $p_+(p, x)$ be the number of singular edges of the local coloring α on the positive side of x, not in x. We have the

Lemma 3. *If x is an elementary cycle*

$$\psi'(\alpha)[x] = \sum_p p_+(p, x).$$

Proof. We can assume that the vertices supporting x are $p_1 \cdots p_n$, with edges $p_n p_1, p_1 p_2, \ldots, p_{n-1} p_n$. Let D_i be the triangle on the positive side of x containing edge $p_{i-1} p_i$. (We take $p_0 = p_n$) $\psi'(\alpha)(x)$ is computed by letting D_1 be $+1$, determining the sign of D_2 using the local coloring around vertex p_1, and continuing. We can easily convince ourself that the assignment on D_i is equal to the assignment on D_{i+1} iff $p_+(p_i, x)$ is 0. Adding around the curve x gives the result. Q.E.D.

Definition. If α is a local coloring, the Stieffel–Whitney class of α is the Z_2 chain consisting of all the singular edges of α. We write it $\mathrm{SW}(\alpha)$.

We claim that $\mathrm{SW}(\alpha)$ is actually a cycle, and so represents a class of $H_1(M; Z_2)$. Lemma 5 of [2] says that the number of nonsingular edges colored p is congruent to the degree of p. Thus, the number of singular edges at p is even.

If α is induced by a heawood coloring, let H be the 2-chain consisting of all triangles labeled $+$. We have $\partial H = \mathrm{SW}(\alpha)$, so $\mathrm{SW}(\alpha) = 0$ in $H_1(M; Z_2)$ if α is induced by a heawood coloring. The converse is true as well, and follows from the next theorem.

THEOREM 4. *Let α be a local coloring, $x \in H_1(M; Z_2)$. Then*

$$\psi'(\alpha)[x] = \mathrm{SW}(\alpha) \cap x.$$

Proof. Since both sides are linear, it suffices to prove it for x an elementary cycle, since such x generate $H_1(M; Z_2)$. ψ' has been only defined on $H_1(M; Z)$, but we compose it with $H_1(M; Z) \to H_1(M; Z_2)$ and call the result ψ'. From the lemma,

$\psi'(\alpha)(X) = \sum_p p_+(p, x)$. We shall show that $\mathrm{SW}(\alpha) \cap x = \sum U_p$, and $U_p = p_+(p, x)$, thus concluding the proof.

To evaluate $\mathrm{SW}(\alpha) \cap X$, we must make X and $\mathrm{SW}(\alpha)$ transversal, so we take the edges of X, and push them off the edges of the triangulation towards the positive side, keeping the end points fixed. Once this is done, the number of singular edges on either side of this new x' is $p_+(p, x)$. We conclude our proof by remarking that if X is an elementary cycle, if Y is any cycle, X and Y transversal, then $X \cap Y = \sum_p (X \cap Y)_p$, where $(X \cap Y)_p$ is the number of edges of Y on one side of X at p.

Q.E.D.

In case M is an even triangulation, and α is the local 3-coloring, we can give more explicit formulas. Suppose X is any cycle of M. Put $p_{1/2}(X)_p$, the semidegree of x, as the number of edges of M at p which lie between x_1 and x_2, x_3 and x_4 or $\cdots$ x_{2n-1} and x_{2n}. Here $x_1 \cdots x_{2n}$ are the vertices of x around the point p. Since $\deg(p)$ is even, this is the same as the number of edges between x_2 and x_3 or x_4 and x_5, etc. We have

$$\psi'(\alpha)(x) = \sum_p p_{1/2}(x)_p .$$

3. Kempe Cycles

This section concerns ways of obtaining one local coloring from another. We begin with the original definition for four colorings. Suppose α and β are both four colorings. If there is a region R such that outside R, α is identical to β, and in R, $\alpha = \sigma\beta$ for some permutation σ

of $\partial\Delta^3$, then we call R a Kempe region of α. Since $\alpha = \beta$ on ∂R, we have that σ fixes the colors of ∂R. In order for α to be different from β, we need that ∂R is 2-colored by α and β (say with colors 1, 2), and the permutation σ is (3, 4).

If we've an edge coloring, the equivalent notion to a Kempe region is a region R whose boundary edges are the same color. To get a new edge coloring, we permute the 2 colors used inside R.

For a heawood coloring, we have a Kempe region as a region R such that the sum of the triangles in R around any point of the boundary is 0 mod 3. We get a new heawood coloring by changing the signs of the triangles of R. If the heawood coloring were induced by a 4 coloring, any Kempe region for the 4 coloring would give such a region but the converse isn't necessarily true. If we look at this change locally, we see that ∂R is a cycle which is locally 2-colored. This gives rise to the following definition.

DEFINITION. A Kempe Cycle of a local coloring α is a Z_2 cycle which is locally 2-colored.

In this definition γ is locally 2-colored by f_p at a vertex p means that $f_p[st(p) \cap \gamma]$ contains at most 2 vertices of $\partial\Delta^3$.

We have the following basic fact: if α is a local coloring of M, γ a Kempe cycle of α, then we can form a labeling of edges of M by calling an edge singular iff it doesn't lie in γ and is a singular edge of α, or it lies in γ and is a nonsingular edge of α. This labeling then comes from a new local coloring β, called the result of changing α along γ. To show this we only need to study the change locally. We are reduced to showing the following.

LEMMA 5. *Let $f: S^1 \to \partial\Delta^2$ be a coloring. If T is an even number of vertices of S^1 which are all colored alike by f, then changing f along T gives a new map g.*

Proof. An easy induction on the number of elements of T.

From the above discussion it should be clear that the relation of Kempe Cycles is implied by the three relations of Kempe regions. The converse is not true. We first observe the following formula for the Stieffel–Whitney Class of β:

$$\mathrm{SW}(\beta) = \mathrm{SW}(\alpha) + \gamma. \tag{*}$$

From this, we see that if R is a Kempe region (for either four coloring, edge coloring, or heawood coloring), then ∂R is a Kempe cycle. If α is related to β by changing α in R, then α is related to β by changing along ∂R (as local colorings). The cycle ∂R is zero in $H_1(M, Z_2)$, $\mathrm{SW}(\alpha) = \mathrm{SW}(\beta)$. Thus if two local colorings have different Stieffel–Whitney Classes, there can be no Kempe region R relating them.

We observe that if α and β are heawood colorings such that there is a Kempe cycle γ with β the result of changing α along γ, then there is a Kempe region R relating α and β. Indeed, since $\mathrm{SW}(\alpha) = 0 = \mathrm{SW}(\beta)$ from (*) we get $\gamma = 0$ in $H_1(M; Z_2)$. Let R be a region such that $\partial R = \gamma$. This is the desired Kempe region. Q.E.D.

The main result for four colorings on the sphere was that any two four colorings of an even triangulation were related by a sequence of Kempe cycles. This is not true for local colorings of an even triangulation of an arbitrary nonsimply connected manifold. For instance, one can embed the 1-skeleton of the 6 simplex on the torus so that it is the 1-skeleton of a triangulation T. This triangulation is even (6 triangles around each vertex) and has 4 equivalence classes of local colorings. In Fig. 1 we have an example of a local coloring α of this triangulation such that there is no local coloring β related to α whose nonsingular edges are also nonsingular edges of α. The possibility of this happening was the basis of the proof on the sphere.

This triangulation does not have, however, a map $T \to \Delta^2$. If we restrict ourselves to the case of an M with a map $M \to \Delta^2$, then all examples considered have one equivalence class.

If we try to copy the old proof, we end up with a Z_2 cycle composed of nonsingular edges which is a local 2 coloring at all but 1 vertex. On the sphere, this implies that it is a Kempe cycle. Even if M has a map to Δ^2, there are examples of local colorings of nonsimply connected M which have no Kempe cycles consisting of nonsingular edges. There can be "singularities." If we change a local coloring along a Z_2 cycle with a singularity, we get a local coloring with a singularity. That is an assignment of edges into two classes (singular and "nonsingular") such that at all but perhaps 1 vertex, this assignment is induced by a four coloring. Such things do exist, but they are phenomena of purely local colorings. For heawood colorings we have the following.

THEOREM 6. *Suppose α is an assignment of $+1$ or -1 to the triangles of a triangulation T of an arbitrary 2-manifold. If at all but 1 vertex p the sum of the assignments around a vertex is zero* mod 3, *then α is a heawood coloring.*

Proof. We shall give two proofs. For the first one, we must show that the sum of the values around p is zero mod 3. This follows from the sequence of equations

$$\sum_{p\in\Delta} \alpha(\Delta) = \sum_{\text{all } q} \Big(\sum_{q\in\Delta} \alpha(\Delta)\Big) = 3 \sum_{\text{all } \Delta} \alpha(\Delta) = 0$$

where $\alpha(\Delta)$ is the assignment on triangle Δ.

Our second proof is just the first proof in the light of our theory. Remove the vertex p from M. We have the map $\psi(\alpha)\colon \Pi_1(M/p) \to S(4)$. If γ is the path around p, we want to show that $\psi(\alpha)[\gamma] =$ identity. Since α is a heawood coloring of p, we have $\psi(\alpha)[\gamma] \in A(4)$ for γ a generator of $\Pi_1(M)$. Now in $\Pi_1(M\backslash p)$, γ is a product of commutators of generators so we have $\psi(\alpha)[\gamma] \in$ commutator subgroup of $A(4) \approx Z_2 \oplus Z_2$. We see $\psi(\alpha)[\gamma]$ fixes a vertex (p), but only the identity in $Z_2 \oplus Z_2$ has a fixed point. Thus $\psi(\alpha)[\gamma]$ is the identity.

We have a similar theorem for edge colorings.

THEOREM 7. *Let α be an assignment of colors a, b, or c to the edges of M such that on all but perhaps one triangle D of M, all three colors occur. Then α is an edge coloring.*

Proof. We again indicate two proofs. If we let a, b and c be the non-zero element of $Z_2 \oplus Z_2$, then we have the following simple fact: if x, y, and z are chosen from a, b, and c (not necessarily distinct), and $x + y + z = 0$ in $Z_2 \oplus Z_2$, then x, y, and z are distinct. Our theorem now follows from the equations below.

$$\sum_{e\in D} \alpha(e) = \sum_{\text{all } \Delta} \sum_{e\in\Delta} \alpha(e) = 2 \sum_{\text{all } e} \alpha(e) = 0.$$

The other proof is to measure $\psi(\alpha)[\partial D]$. We see it's the product of commutators, but the commutator subgroup of $Z_2 \oplus Z_2$ is the identity. By passing to homology, one can see that the two proofs are really the same. Q.E.D.

4. DEGREE AND NUMBER

In this section we shall discuss some elementary facts about the degrees of colorings and some empirical facts about the number of colorings.

If f is a four coloring of M, then the degree of f is well defined. By $\bar{f}(\Delta)$

let us denote the assignment on Δ by f considered as a heawood coloring. We have the following elementary formulas:

(1) $\deg f = \frac{1}{4} \sum_{\Delta} \bar{f}(\Delta)$;
(2) $\deg f = \frac{1}{4} \sum_{p} \deg(f \mid \ln k(p))$.

In formula (2), we have $f \mid \ln k(p) : \ln k(p) \to \partial\Delta^2$ as a map between 1 spheres, so we get a well defined integer. M of course must be oriented.

Both sums on the right-hand side are well defined for a heawood coloring. Thus we can say that if the sum of the assignments to the triangles of a heawood coloring is $4m + 2$, then the heawood coloring isn't induced by a four coloring.

If M has a map to Δ^2, we have some restrictions on the degree of a four coloring f. From the diagram in the proof of Theorem 1 in [F], we conclude that $6 \mid \deg(f)$. This follows from the fact that the degree of the projection $\Delta^2 \times \partial\Delta^3 \to \partial\Delta^3$ is 6.

For local colorings, we have not found any invariant. That is, we have found no way of assigning to each local coloring an element of a group which is preserved under equivalence by Kempe cycles. It is of interest to find some invariants of local coloring.

If we consider the number of local colorings, the situation is quite confusing. We can only state the following facts, based on little empirical data.

(1) If M has a map to Δ^2, and is irreducible then the number of local colorings is $= 1 \bmod 4$.

(2) If M is any even triangulation, the number of local colorings with a fixed nonzero Stieffel–Whitney class is 0 mod 4.

5. Existence of Kempe Cycles

On the sphere it is easy to find triangulations which have colorings with no Kempe cycles. The smallest nontrivial example has 8 vertices. On a nonsimply connected surface we shall show every four coloring has a Kempe cycle. For even triangulations, it may be that every local coloring has a Kempe cycle.

Theorem 8. *Every four coloring of a nonsimply connected manifold has a Kempe cycle.*

Proof. Let $f: M \rightarrow \partial\Delta^3$ be a four coloring. Pick a pair of vertices of $\partial\Delta^3$, say 0, 1. Suppose that there is no 01 cycle. We shall show that there are 23 cycles, where 2, 3 are the other two vertices of $\partial\Delta^3$. Let L be the subcomplex spanned by the 01 cycles, and N the subcomplex spanned by the 23. Since L has no cycles, L has the homotopy type of several points. Now $M \backslash L$ has the homotype type of N, and L is just points, so $\Pi_1(M \backslash L) = \Pi_1(N)$ is not zero. Thus N contains a cycle, and we're done.

In fact, we've shown that every four coloring has at least 6 Kempe cycles. If there is no 01 cycle, then since $\Pi_1(M) \rightarrow \Pi_1(M/L)$ injects, there are at least two 23 cycles.

6. Relations to the Four Color Problem

In this section we shall prove the results stated in [1]. Familiarity with Ref. [1] is assumed.

THEOREM 9. *$R \# S$ have an edge coloring iff R and S both do.*

Proof. If R and S do, it's trivial. The other direction follows from Theorem 7.

THEOREM 10. *If every even surface has an edge coloring, then G4CC is true.*

Proof. Take a triangulation X of the torus which has two odd vertices which are adjacent, and all the other vertices even. Let (D) be a triangle of X containing these two odd vertices. If we form a connected sum of a triangulation S with X, adding X along D, then the resulting triangulation has at most as many odd vertices as x. Moreover, the pairity of the two vertices of S which were joined to the old vertices of X is changed in X. Thus we can, by adding enough copies of X, get even triangulation $T = S \# X \# X \cdots \# X$. If T has an edge coloring, then by the last theorem, S has one too.

The rest of the results of [1] follow easily from Section III.

7. Involutions and the Peterson Graph

There is a triangulation P of the projective plane which has no local coloring. The dual of P is known in the literature as the Peterson graph, and is the only example of a cubic graph with no edge coloring. We define P as the quotient of the icosahedron by the antipodal map. Thus P has 6 vertices, each of degree 5.

Let G be a triangulation of the sphere and let σ be a fixed point free involution on G. G/σ is a triangulation of the projective plane. If α is any local coloring of G/σ, it lifts to a local coloring $\hat{\alpha}$ on G. Since every local coloring of the sphere is a 4 coloring, $\hat{\alpha}$ is a 4 coloring. We claim that the degree of $\hat{\alpha}$ is even. Consider ψ_α: $\Pi_1(G/\sigma) \to S(4)$. Since $\Pi_1(G/\sigma)$ is Z_2, $\mathrm{Im}(\psi_\alpha)\backslash 1$ is either (12) or (12)(34), up to conjugation. If $\tau = \mathrm{Im}(\psi_\alpha)\backslash 1$ then we have the relation $\alpha\sigma = \tau\hat{\alpha}$.

There are two cases. First suppose $\tau = (12)(34)$. Then the orientations of Δ and $\sigma\Delta$ disagree, but both are assigned $+1$ (or -1) by $\hat{\alpha}$. Thus the orientations assigned by $\hat{\alpha}$ in the heawood coloring to Δ and $\sigma\Delta$ are opposite, so the sum of the orientations assigned by α to the triangles is 0. This is 4 times the degree, so the degree is 0.

Next suppose $\tau = (12)$. Consider the set S of triangles labeled (123) by α. Then from $\hat{\alpha}\sigma = \tau\hat{\alpha}$ we see that σ is a fixed point free involution on S. Since the degree mod 2 is just the number of elements of S, we have deg $\hat{\alpha}$ is even. Summarizing this, we have

THEOREM 11. *Let σ be a fixed point free involution on a triangulation G of the sphere. If α is*

(1) *a local coloring of G/σ then* deg $\hat{\alpha}$ *is even*

(2) *a heawood coloring of G/σ then* deg $\hat{\alpha}$ *is* 0.

If we consider the icosahedron, we can see that it has 10 distinct four colorings, each of degree 1. Combining this with the theorem, we see the Peterson triangulation has no local coloring. It would be interesting to have a reason for why the icosahedron has all its colorings of degree 1. We make the conjecture:

> The only irreducible triangulation of the sphere with all four colorings of odd degree and a fixed point free involution is the icosahedron.

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